LPC State HW. 1
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a)
$a(b+c) = ab + ac$ (2)
let $b = c = 0$
$a(0+0) = a0 + a0$
$a(0) = a0 + a0$ (3)
$a0 = aa\bar{a} + aa\bar{a}$ (8)
$a0 = a\bar{a} + a\bar{a}$ (lemma 1 : $aa = a$)
$a0 = 0 + 0$ (8)
$a0 = 0$ (3)

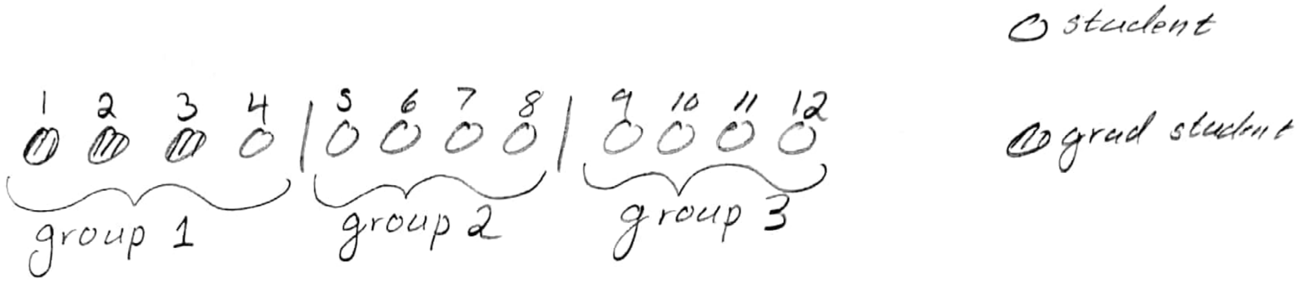
b)

c) start from result b.
$a + ab = a$
$aa + ab = a$ (lemma 1)
$a(a+b) = a$ (6)

Lemma 1
$A1 = A$ (7)
$A(A+\bar{A}) = A$ (4)
$AA + A\bar{A} = A$ (6)
$AA + 0 = A$ (8)
$AA = A$ (3)

P2.

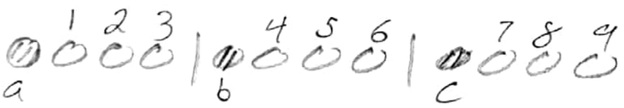
a) 12 students are divided into 3 groups.



first of all, there are 12! total permutations of students, as arranged above. However, each group will see 4! permutations of the same students and there are 3! identical groupings, just in different orders. Therefore, the number of unique groupings is.

$$\frac{12!}{(4!)^3\,3!} = 5775$$

b) if each group has 1 grad student, the picture looks like



so we have a's group, b's group, and c's group. This effectively labels the groups so we no longer have to divide out ~~the~~ the factor of 3! from group ordering. What remains is

$$\frac{9!}{(3!)^3} = 1680$$

P3. consider a group of p people and a calendar of d days, and represent a combination of birthdays by

$\sqcup\ \underline{3}\ \underline{4}\ \underline{7}\ \sqcup$ (eg. five people w/ a calendar of 7 days)

first, lets find the number of ways to fill the slots that result in exactly 2 people sharing a birthday. Let's assign the lucky guys first. there are d birthdays that can be shared and $\frac{p(p-1)}{2}$ ways to assign them into the slots. for example,

$\sqcup\ \underline{3}\ \sqcup\ \underline{3}\ \sqcup$ (slots 2 and 4 share bday)

the remaining slots (in this case, (1,3,5)) can be filled with all permutations of all combinations of the remaining $d-1$ days. In total, this is

$$\binom{d-1}{p-2}\cdot(p-2)!$$

combining this with the previous factors gives

$$d\cdot\frac{p(p-1)}{2}\binom{d-1}{p-2}(p-2)!$$

$$= d\cdot\frac{p(p-1)}{2}\frac{(d-1)!}{(d-p+1)!}$$

the total number of ways of filling the slots, regardless of sharing, is d^p, so the probability of a random group having 1 shared birthday is

$$P(p,d)=\frac{p\,(p-1)(d-1)!}{2d^{p-1}(d-p+1)!}$$

finally, for $d=365$, there is no value of p yielding P larger than 0.5. P reaches a maximum of ≈ 0.38 at $p=28$. This differs from the ordinary birthday problem because we are asking for <u>exactly</u> 1 pair instead of any number of pairs, triplets, and so on.

4. Bayes' Theorem relates the conditional probabilities

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

now, in the given problem, define the following 3 events

D_1: car behind door 1
D_3: car behind door 3
M_2: Monty opens door 2

we know that monty has opened door 2 so we want the conditional probabilities

$$P(D_1|M_2) \text{ and } P(D_3|M_2)$$

so, using Bayes' theorem,

$$P(D_1|M_2) = \frac{P(M_2|D_1)\,P(D_1)}{P(M_2)}$$

we know, first of all, that $P(D_1) = P(D_3) = 1/3$. We also know that if the car is behind Door 1, monty could open door 2 or 3 with equal odds. so $P(M_2|D_1) = 1/2$. Finally, since monty always could open either of the two doors not picked $P(M_2) = 1/2$.

$$\Rightarrow P(D_1|M_2) = \frac{1/2 \cdot 1/3}{1/2} = 1/3$$

for $P(D_3|M_2)$, the difference is that monty cannot open door 3 so $P(M_2|D_3) = 1$.

$$\Rightarrow P(D_3|M_2) = \frac{1 \cdot 1/3}{1/2} = 2/3$$

5. a) $t = a\sum_{i=1}^{n} x_i$

where, x_i random variable with pdf $U(x,\mu,\sigma) = \frac{1}{2}$, $|x|<1$

0 , otherwise

since the x_i are uncorrelated, we can use the Bienaymé formula to calculate the variance of t from the variance of x_i.

$$Var(x_i) = E(x_i^2) - E(x)^2$$

$$E(x_i)^2 = 0 \text{ , (even pdf)}$$

$$E(x_i^2) = \int_{-1}^{1} \tfrac{1}{2}x_i^2 = \tfrac{1}{6}x_i^3\Big|_{-1}^{1} = 1/3$$

the a^2 comes from the rule where multiplying a variable by a constant increases its variance by that factor squared.

$$\Rightarrow Var(x_i) = 1/3$$

$$\Rightarrow Var(t) = a^2\sum_{i=1}^{n} Var(x_i) = \frac{a^2 n}{3}$$

since this is set to 1, we can solve for a.

$$\frac{a^2 n}{3} = 1 \Rightarrow a = \sqrt{3/n}$$

b) The pdf of the sum of 2 random variables is the convolution of the 2 variables pdfs. so for 2 x's

$$U_2 = \int_{-\infty}^{\infty} U(x')U(x-x')\,dx'$$

at this point limits get tricky, so introduce the step function to help.

$$H(x) = \begin{cases} 1 , & x \geq 0 \\ 0 , & x < 0 \end{cases}$$

with this, U becomes

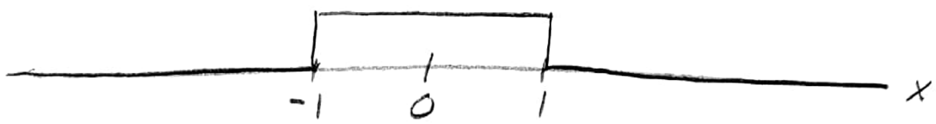
$$U(x) = \tfrac{1}{2}H(1+x)H(1-x)$$

5. cont)

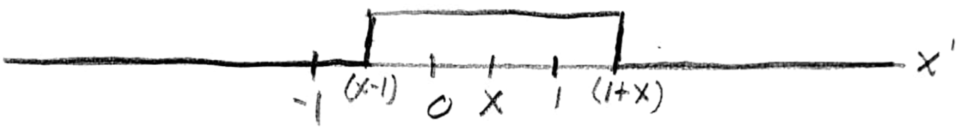
putting this into u_2 gives

$$u_2 = \int_{-\infty}^{\infty} \tfrac{1}{4} \underbrace{H(1+x')\,H(1-x')}_{A}\ \underbrace{H(1-x'+x)\,H(1+x'-x)}_{B}\, dx'$$

A



B

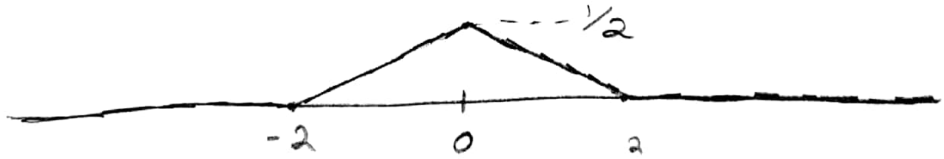


A × B

looking at the visual effects of the product of step functions, it is straight forward to see that the result of the convolution is.

$$u_2 = \frac{1}{2}\left(1 - \frac{|x|}{2}\right)$$

u_2:



While this approach is ok for the sum of 2 variables, it doesn't lend itself well to the sum of n variables. So lets instead use the random variable theorem.

$$\boxed{\begin{gathered} p(t) = \int dz_1 \cdots \int dz_n\, \delta(t - g(z_1, \cdots, z_n))\, p(z_1, \ldots z_2) \\ t \equiv g(z_1, \ldots z_n) \end{gathered}}$$

5, cont)

in our case,

$$g = a\sum_{i=1}^{n} x_i$$

$$p(x_1,\dots,x_n) = \prod_{i=1}^{n} p(x_i) = \begin{cases} \frac{1}{2^n}, & \forall x_i \in (-1,1) \\ 0, & \text{otherwise} \end{cases}$$

so,

$$p(t) = \int_{-1}^{1} dx_1 \cdots \int_{-1}^{1} dx_n \, \delta\left(t - a\sum_{i=1}^{n} x_i\right) \frac{1}{2^n}$$

next, use the representation of the delta function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} d\omega$$

$$\Rightarrow p(t) = \int_{-1}^{1} dx_1 \cdots \int_{-1}^{1} dx_n \frac{1}{2\pi 2^n}\left(\int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \prod_{i=1}^{n} e^{-ia\omega x_i}\right)$$

and recklessly shuffling order of integration gives

$$p(t) = \frac{1}{2\pi 2^n}\int_{-\infty}^{\infty} d\omega \, e^{i\omega t}\left(\int_{-1}^{1} dx_1 \cdots \int_{-1}^{1} dx_n \prod_{i=1}^{n} e^{-ia\omega x_i}\right)$$

$$= \frac{1}{2\pi 2^n}\int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \prod_{i=1}^{n} \int_{-1}^{1} dx_i \, e^{-ia\omega x_i}$$

and the domain is symmetric,

since $e^{-ix} = \cos(x) - i\sin(x)$

$$\int_{-1}^{1} dx_i \cos(a\omega x_i) = \frac{2\sin(a\omega)}{a\omega}$$

$$\Rightarrow p(t) = \frac{1}{2\pi \cancel{2^n}}\int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \, \cancel{2^n}\left[\frac{\sin(a\omega)}{a\omega}\right]^n$$

$$\boxed{p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, e^{i\omega t}\left[\frac{\sin(a\omega)}{a\omega}\right]^n}$$

5. cont)

c) show that for
$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\boxed{\left(\frac{\sin(a\omega)}{a\omega}\right)^n} d\omega$$

that
$$\lim_{n\to+\infty} p(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}$$

expanding the boxed term at $n=+\infty$,
$$\left(\frac{\sin(a\omega)}{a\omega}\right)^n \rightarrow e^{-\frac{\omega^2}{2}} + O\left(\tfrac{1}{n}\right)$$

plugging this back in,
$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t - \omega^2/2}\, d\omega$$

. now complete the square in the exponent
$$i\omega t - \omega^2/2 = -\tfrac{1}{2}(\omega^2 - 2i\omega t) = -\tfrac{1}{2}\left[(\omega - it)^2 + t^2\right]$$
$$\Rightarrow p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\omega-it)^2} e^{-\frac{1}{2}t^2} d\omega$$
$$= \frac{1}{2\pi} e^{-\frac{1}{2}t^2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\omega-it)^2} d\omega}_{\sqrt{2\pi}}$$

$$\Rightarrow p(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$